

Vibrations of a Free-Free Beam

The bending vibrations of a beam are described by the following equation:

$$EI \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$



E, I, ρ, A are respectively the Young Modulus, second moment of area of the cross section, density and cross section area of the beam. L is the length of the beam. The solution of Eq. (1) can be written as a standing wave¹ $y(x,t) = w(x)u(t)$, separating the spatial and temporal component. This leads to the following characteristic equation that relates the circular frequency ω to the wavenumber k :

$$\omega^2 = \frac{EI}{\rho A} k^4 \quad (2)$$

The spatial part can be written as:

$$w(x) = C_1 \sin(kx) + C_2 \cos(kx) + C_3 \sinh(kx) + C_4 \cosh(kx) \quad (3)$$

For a Free-Free Beam the boundary conditions are (vanishing of force and moment):

$$\left\{ \begin{array}{l} w''(0) = 0 \\ w'''(0) = 0 \\ w''(L) = 0 \\ w'''(L) = 0 \end{array} \right. \text{ to get } \left\{ \begin{array}{l} -C_2 + C_4 = 0 \\ -C_1 + C_3 = 0 \\ -C_1 \sin(kL) - C_2 \cos(kL) + C_3 \sinh(kL) + C_4 \cosh(kL) = 0 \\ -C_1 \cos(kL) + C_2 \sin(kL) + C_3 \cosh(kL) + C_4 \sinh(kL) = 0 \end{array} \right.$$

¹ A standing wave or stationary wave is a wave 'frozen' in the space and vibrating in time. It result by the sum of two identical waves travelling in opposite directions:

$$y(x,t) = y_+(x,t) + y_-(x,t) = Y_0 \sin(\omega t - kx) + Y_0 \sin(\omega t + kx) = 2Y_0 \sin(kx) \cos(\omega t) = w(x)u(t)$$

Using the first two equations the 3rd and 4th can be arranged in matrix form:

$$\begin{bmatrix} \sinh(kL) - \sin(kL) & \cosh(kL) - \cos(kL) \\ \cosh(kL) - \cos(kL) & \sin(kL) + \sinh(kL) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4)$$

For a non trivial solution the determinant of the matrix has to vanish to get:

$$\cosh(kL)\cos(kL) = 1 \quad (5)$$

The transcendental Eq. (5) has infinite solutions, it can be solved numerically, the first five values are reported here:

Mode order n	$k_n L$
0	0
1	4.7300
2	7.8532
3	10.9956
4	14.1371
5	17.2787

Putting these values back in Eq. (5) gives the modeshapes corresponding to the natural frequencies ω_n that can be calculated from the characteristic Eq. (2). The mode shapes are given by the following:

$$w_n(x) = [\sinh(k_n x) + \sin(k_n x)] + \frac{\sin(k_n L) - \sinh(k_n L)}{\cosh(k_n L) - \cos(k_n L)} [\cosh(k_n x) + \cos(k_n x)] \quad (6)$$

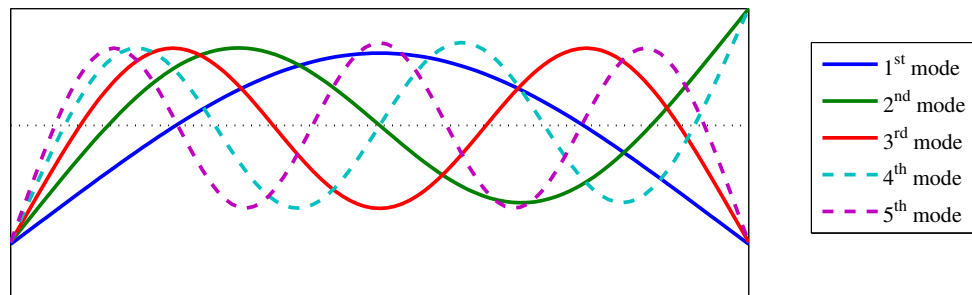


Figure 1. First 5 mode shapes for a free-free beam

The velocity of the bending waves in the beam, also called phase velocity, is given by

$$c_B = \frac{\omega}{k} = \sqrt[4]{\frac{EI}{\rho A}} \sqrt{\omega} .$$

It shows that the velocity depend on the frequency. A generic wave

travelling in the beam can be described in terms of several harmonics as given by the Fourier analysis, components at higher frequency travel faster creating then a continuous distortion, for this reason they are called dispersive waves.

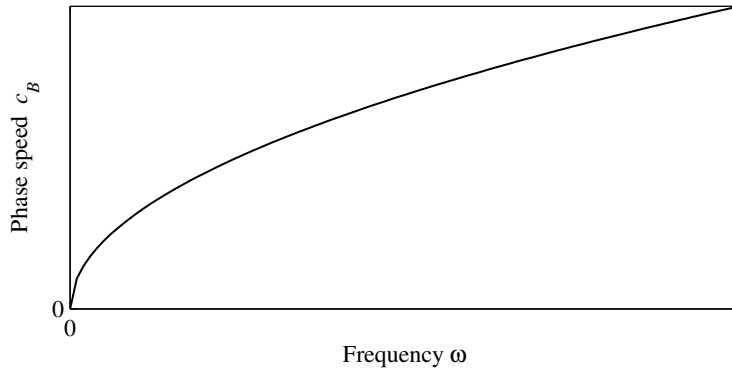


Figure 2. Phase speed of bending waves

It is interesting to notice that for a Clamped-Clamped beam, the boundary conditions are: (vanishing of displacement and slope):

$$\begin{cases} w(0) = 0 \\ w'(0) = 0 \\ w(L) = 0 \\ w'(L) = 0 \end{cases} \text{ to get } \begin{cases} C_2 + C_4 = 0 \\ C_1 + C_3 = 0 \\ C_1 \sin(kL) + C_2 \cos(kL) + C_3 \sinh(kL) + C_4 \cosh(kL) = 0 \\ C_1 \cos(kL) - C_2 \sin(kL) + C_3 \cosh(kL) + C_4 \sinh(kL) = 0 \end{cases}$$

Using the first two equations the 3rd and 4th can be arranged in matrix form:

$$\begin{bmatrix} \sinh(kL) - \sin(kL) & \cosh(kL) - \cos(kL) \\ \cosh(kL) - \cos(kL) & \sin(kL) + \sinh(kL) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

That is the same as before, then the resonance frequencies are the same as in the case of a free-free beam except that in this case we do not have the two rigid body modes (translation and rotation at $\omega = 0$) since it is not allowed by the boundary conditions.

Chladni² patterns

It is possible in a laboratory experiment to visualize the nodes of a vibrating beam (nodal lines in this case since the beam has a real width) by sprinkling sand on it: the sand is thrown off the moving regions and piles up at the nodes. The beam is excited with a shaker at exactly his natural frequencies. In this case (low modal coupling) is reasonable to assume that at resonance the deformation shape is mainly given by the mode shape corresponding to the resonant frequency excited. The results for a free-free beam can be seen in the following video:



<http://www.youtube.com/watch?v=XkmgMkDKAyU>

The free-free conditions were simulated suspending the beam with springs introducing an extra natural frequency, reasonably lower than the first resonance in bending vibration. Nevertheless the suspension system gave some troubles in visualising the two nodes at the first resonance. Other mode shapes can be seen quite clearly and the resonance frequencies values are not too far from the theoretical results. The data of the beam are:

$$L = 1.275 \text{ m} \quad A = h \times b = 0.01 \times 0.075 \text{ m} \quad I = \frac{bh^3}{12}$$

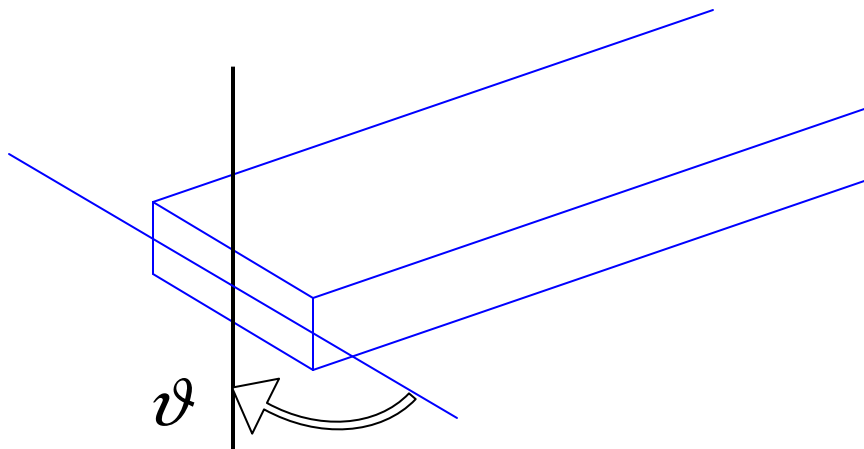
$$\rho = 7800 \text{ Kg m}^{-3} \quad E = 2.1 \times 10^{11} \text{ Nm}^{-1} \quad \nu = 0.3$$

² Ernst Florens Friedrich Chladni was a German physicist. Chladni's technique, first published in 1787 in his book, "Discoveries in the Theory of Sound", consisted of drawing a bow over a piece of metal whose surface was lightly covered with sand. The plate was bowed until it reached resonance and the sand formed a pattern showing the nodal regions.

$f_n = \frac{\omega_n}{2\pi}$	Theoretical [Hz]	Experimental [Hz]
$n=1$	32.80	32.25
$n=2$	90.44	88.50
$n=3$	177.30	173.50
$n=4$	293.08	287.50
$n=5$	437.82	430.00

Table 1. First five natural frequencies in bending vibration

Since the beam in this case is a real piece of steel, there are also longitudinal, in plane and torsional vibrations. In this experiment the shaker was exciting the beam vertically at one corner so that it is possible to see also torsional modes. The values for the torsional vibration can be calculated considering the torsional vibration for a beam of no-circular cross section.



The variation of angular orientation $\vartheta(x, t)$ for the cross section of the beam is described by the following torsional vibration equation:

$$\frac{\partial^2 \vartheta}{\partial t^2} - \frac{G\gamma}{\rho J_p} \frac{\partial^2 \vartheta}{\partial x^2} = 0 \quad (7)$$

$G = E/2(1+\nu)$ is the shear modulus, ν the Poisson ratio, γ is a torsional constant that for a rectangular cross section is $\gamma \approx bh^3 \left[\frac{1}{3} - 0.21 \frac{h}{b} \left(1 - \frac{h^4}{12b^4} \right) \right]$. $J_p = \frac{bh}{12}(b^2 + h^2)$ is the polar moment of area of the cross section. The solution of Eq. (7) can be written as a standing wave:

$$\vartheta(x,t) = \theta(x)u(t) \quad (8)$$

The spatial part can be written as:

$$\theta(x) = A_1 \sin(k_T x) + A_2 \cos(k_T x) \quad (9)$$

With $k_T = \frac{\omega}{c_T}$ and $c_T = \sqrt{\frac{G\gamma}{\rho J_p}}$ is the phase velocity, in this case is constant with frequency and the waves are not dispersive. For a Free-Free Beam the boundary conditions are: (vanishing of the moment):

$$\begin{cases} \theta'(0) = 0 \\ \theta'(L) = 0 \end{cases} \quad \text{to get} \quad \begin{cases} A_1 = 0 \\ \sin(k_T L) = 0 \end{cases} \rightarrow k_{T,n} = \frac{n\pi}{L}$$

With the values of the torsional wavenumber $k_{T,n}$ we get the mode shapes:

$$\theta_n(x) = \cos(k_{T,n} L) \quad (10)$$

And the natural frequencies are given by

$$\omega_{T,n} = \frac{n\pi}{L} c_T \quad (11)$$

With the data of the experimental beam we get the first torsional mode is at

$$f_{T,1} = \frac{c_T}{2L} = 319.24 \text{ Hz, not too far from the experimental result of 315 Hz.}$$